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LETTER TO THE EDITOR

New class of conditionally exactly solvable potentials in quantum mechanics

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Abstract. Motivated by an idea of Dutra, we obtain a new class of one-dimensional conditionally exactly solvable potentials for which the entire spectra can be obtained in an algebraic manner provided one of the potential parameters is assigned a fixed negative value. It is shown that using shape-invariant potentials as input, one may generate different classes of such potentials even in more than one dimension. We also illustrate that WKB and supersymmetry inspired WKB methods provide very good approximations for these potentials with the latter doing comparatively better.

Interest in obtaining exact solutions for non-relativistic quantum potentials has been intense in recent times. One of the major reasons behind this is the fact that knowledge of exact solutions can be used as a basis to perform a variety of perturbative as well as non-perturbative approximation methods for the non-exact potentials which occur in many branches of physics. These investigations have been further stimulated by the observation of a nice connection between the factorization method [1–4] based on supersymmetric quantum mechanics (SUSYQM) [5–7] and second-order differential equations. A new class of potentials, known as quasi-exactly solvable (QES) potentials [8–11], has also been discovered for which only a finite number of eigenstates are known analytically under certain constraint conditions among the potential parameters while the remaining ones have to be obtained numerically. Such analytical solutions are useful in testing the accuracy of the eigenvalues obtained by numerical integration methods. Besides, they are also known to reveal some interesting group theoretic structure [9, 10].

Very recently, Dutra [12] has discovered a different class of quantum potentials known as conditionally exactly soluble (CES) potentials for which exact eigenvalues and eigenfunctions of all quantum states can be obtained provided one of the potential parameters is assigned a fixed negative value through a mapping procedure. In his approach, only simple power-law-type transformations have been used as the mapping function. At this stage, it is quite natural to enquire whether one can generate different classes of CES potentials invoking more complex mapping functions. The purpose of this letter is to demonstrate that with the help of a shape-invariant class of exactly solvable potentials [4] and using transcendental mapping functions, one may obtain new CES potentials which are quite different from the standard quantum potentials. Here we present the steps for obtaining two one-dimensional CES potentials. However, it is clear that similar potentials in higher dimension can also be constructed following our procedure. For the sake of completeness, we also present numerical comparisons of the eigenvalues obtained from the leading-order WKB and supersymmetry-inspired WKB (SWKB) quantization conditions.

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To obtain one-dimensional CES potentials, we begin with the Schrödinger equation (in units of $\hbar = 2m = 1$)

$$\left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E\psi(x) \tag{1}$$

for a potential $V(x)$. Invoking a transformation of the independent variable

$$x = f(u) \tag{2}$$

and redefining the wavefunction as

$$\psi(x) = \sqrt{f'(u(x))}\chi(u) \tag{3}$$

where the prime denotes differentiation with respect to the variable u, we obtain a transformed Schrödinger equation

$$\left[-\frac{\partial^2}{\partial u^2} + V_{\mathrm{T}}(u)\right]\chi(u) = E_{\mathrm{T}}\chi(u) \tag{4}$$

in which

$$V_{\mathrm{T}}(u) - E_{\mathrm{T}} = (f'(u))^2[V(f(u)) - E] + \Delta V(u) \tag{5}$$

with

$$\Delta V(u) = \left[-\frac{1}{2}\frac{f'''(u)}{f'(u)} + \frac{3}{4}\left(\frac{f''(u)}{f'(u)}\right)^2\right]. \tag{6}$$

One now chooses any exactly solvable (ES) potential as V_{T} and the idea is to find the transformation functions $f(u)$ such that one would have new analytically solvable potentials $V(x)$. Obviously, the non-trivial part is the proper choice of $f(u)$ so that $V(x)$ as well as energy eigenvalues and eigenfunctions can be expressed in a closed form.

For our purpose, we shall use the shape-invariant class of potentials [4] as the input information. As for illustration, we consider the mapping function in (2) as

$$x = f(u) = \log(\sinh u) \qquad \text{or} \qquad \sinh u = \mathrm{e}^x. \tag{7}$$

Obviously, the domain of the variable u is $0 \leqslant u \leqslant \infty$, corresponding to $-\infty \leqslant x \leqslant \infty$. In this case $\Delta V(u)$ turns out to be

$$\Delta V(u) = -\tfrac{1}{4}\,\mathrm{cosech}^2\, u - \tfrac{3}{4} + \tfrac{3}{4}\tanh^2 u. \tag{8}$$

We should mention at this point that two different choices can be made for $V(f(u))$.

Case (a): If we now choose

$$V(f(u)) = A\tanh^2 u - B\tanh u + C\tanh^4 u \qquad C = -\tfrac{3}{4}. \tag{9}$$

equations (5), (8) and (9) lead to

$$V_{\mathrm{T}}(u) = -B\coth u - (E + \tfrac{1}{4})\operatorname{cosech}^2 u \tag{10}$$

and

$$E_{\mathrm{T}} = -A + \tfrac{3}{4} + E. \tag{11}$$

However $V_{\mathrm{T}}(u)$ as given by equation (10) is an exactly solvable potential with well known energy eigenvalues and eigenfunctions. In fact, from [4, 13] we obtain for a potential

$$V_{\mathrm{T}}(u) = -2b\coth u + a(a-1)\operatorname{cosech}^2 u \qquad b > a^2, a > 0 \tag{12}$$

the energy eigenvalues as

$$E_{\mathrm{T}} = -\left(\frac{b}{a+n}\right)^2 - (a+n)^2 \tag{13}$$

and the corresponding eigenfunctions $\chi(u)$ are given by

$$\chi(u) = (y-1)^{-(a+n-\lambda)/2}(y+1)^{-(a+n+\lambda)/2}P_n^{-(a-n+\lambda,-a-n-\lambda)}(y) \tag{14}$$

where $y = \coth u$, $\lambda = b/(n+1)$ and $P_n^{\alpha,\beta}(y)$ are Jacobi polynomials. Comparing equations (10) and (12), we get

$$2b = B \qquad a - \tfrac{1}{2} = \sqrt{-E}. \tag{15}$$

Comparing (11) and (13) and using (15), we obtain

$$A + \varepsilon_n - \tfrac{3}{4} = \left(\frac{B/2}{n+\frac{1}{2}+\sqrt{\varepsilon_n}}\right)^2 + (n + \tfrac{1}{2} + \sqrt{\varepsilon_n})^2 \tag{16}$$

in which $E_n = -\varepsilon_n$, $\varepsilon_n > 0$. This after simple manipulation leads to the cubic equation

$$\begin{aligned}2(n+\tfrac{1}{2})\varepsilon_n^{3/2} &+ [5(n+\tfrac{1}{2})^2 - (A-\tfrac{3}{4})]\varepsilon_n + [4(n+\tfrac{1}{2})^3 - 2(n+\tfrac{1}{2})(A-\tfrac{3}{4})]\sqrt{\varepsilon_n}\\ &+ [(n+\tfrac{1}{2})^4 - (n+\tfrac{1}{2})^2(A-\tfrac{3}{4}) + B^2/4] = 0\end{aligned} \tag{17}$$

from where one obtains ε_n for given A and B. From the three roots we can discard two by demanding that the spectrum must reduce to the standard one for $B = 0$. Once the eigenvalues have been determined, the corresponding eigenfunctions $\psi_n(x)$ are immediately obtained by using equations (3), (7) and (15). We obtain

$$\psi(x) = y^{1/4}(y-1)^{-(c/2-B/4c)}(y+1)^{-(c/2-B/4c)}P_n^{(B/2c-c,-B/2c-c)}(y) \tag{18}$$

where $y = (1+\exp(-2x))$ and $c = n + 1/2 + \sqrt{\varepsilon_n}$.

Equation (17) and (18) give the energy eigenvalue and the eigenfunctions of our first CES potential

$$V(x) = \frac{A}{(1+\mathrm{e}^{-2x})} - \frac{B}{\sqrt{1+\mathrm{e}^{-2x}}} - \frac{3}{4(1+\mathrm{e}^{-2x})^2} \tag{19}$$

which is essentially the same as (9) in terms of the original variable.

Case (b): Unlike (9), one may consider

$$V(f(u)) = A\tanh^2 u - B\,\mathrm{sech}\,u - \tfrac{3}{4}\tanh^4 u. \tag{20}$$

Using (8) and (20) in equation (5), one obtains

$$V_\mathrm{T}(u) = -B\,\mathrm{cosech}\,u\coth u - (E+\tfrac{1}{4})\,\mathrm{cosech}^2 u \tag{21}$$

$$E_\mathrm{T} = E + \tfrac{3}{4} - A. \tag{22}$$

Again $V_\mathrm{T}(u)$ in (21) is an ES potential. It is known that for the potential ($b > a > 0$)

$$V_\mathrm{T}(u) = [a(a+1) + b^2]\,\mathrm{cosech}^2 u - b(2a+1)\coth u\,\mathrm{cosech}\,u \tag{23}$$

the energy eigenvalues and eigenfunctions are given by [4, 13]

$$E_\mathrm{T} = -(a-n)^2 \tag{24}$$

$$\chi_n(u) = (y-1)^{(b-a)/2}(y+1)^{-(b+a)/2}P_n^{(b-a-1/2,b-a-1/2)}(y) \tag{25}$$

where $y = \cosh u$. On comparing equations (23) and (24) with (21) and (22), we obtain ($E_n = -\varepsilon_n$, $\varepsilon_n > 0$)

$$2b = \sqrt{\varepsilon_n + B} + \sqrt{\varepsilon_n - B} \qquad 2a = \sqrt{\varepsilon_n + B} - \sqrt{\varepsilon_n - B} - 1 \tag{26}$$

and the energy eigenvalues for the potential (20) are given in terms of A and B by

$$\sqrt{\varepsilon_n + A - \tfrac{3}{4}} = \tfrac{1}{2}\sqrt{\varepsilon_n + B} - \tfrac{1}{2}\sqrt{\varepsilon_n - B} - (n + \tfrac{1}{2}). \tag{27}$$

From here one again obtains a cubic equation for ε_n and as before one can reject two roots by demanding that the spectrum be reduced to the standard form for $B = 0$. Once the eigenvalues have been determined, the eigenfunctions $\psi_n(x)$ are immediately obtained by using equations (3), (7) and (25). Rewriting (21) in terms of the original variable x, we get our second CES potential in one dimension:

$$V(x) = \frac{A}{1+\mathrm{e}^{-2x}} - \frac{B\mathrm{e}^{-x}}{\sqrt{\mathrm{e}^{-2x}+1}} - \frac{3}{4(1+\mathrm{e}^{-2x})^2}. \tag{28}$$

Our new potentials (19) and (28) are quite different from the standard potentials used in quantum problems. It is interesting to note that, in both cases, the third terms are identical and have the same fixed negative coefficient $(-\frac{3}{4})$. We suspect that this may have a subtle connection to the complete solvability of the Schrödinger equation for these two potentials.

Thus we have obtained two new CES potentials for which the entire eigenspectrum can be obtained analytically.

The SWKB method [14] gives the exact energy values in the lowest order for all shape-invariant potentials [15], while the WKB method in the lowest order is able to give the exact result only for the harmonic oscillator and the Morse potential. It is of interest to enquire as to how good the WKB and the SWKB approximations are in the case of these two potentials.

To this end we have computed the spectrum numerically by using the lowest-order WKB quantization condition

$$\int_{x_1}^{x_2} \sqrt{E_n - V(x)}\,\mathrm{d}x = (n + \tfrac{1}{2})\pi \qquad n = 0, 1, 2, \ldots \tag{29}$$

as well as the lowest-order SWKB condition

$$\int_{a}^{b} \sqrt{E_n - W^2(x)}\,\mathrm{d}x = n\pi \qquad n = 0, 1, 2, \ldots \tag{30}$$

where $W(x) = -\psi_0'(x)/\psi_0(x)$. For the two potentials it is easily seen that

$$W_a(x) = \frac{B}{(1 + 2\sqrt{\varepsilon_0})\sqrt{1 + \mathrm{e}^{-2x}}} - \frac{1}{2(1 + \mathrm{e}^{-2x})} - \sqrt{\varepsilon_0} \tag{31}$$

$$W_b(x) = a + \frac{1}{2(1 + \mathrm{e}^{2x})} - \frac{b}{\sqrt{1 + \mathrm{e}^{2x}}}. \tag{32}$$

The WKB and the SWKB eigenvalues for the two potentials are compared with the corresponding exact answers in tables 1 and 2. The two potentials have also been plotted in figures 1 and 2. From the tables we find that both WKB and the SWKB are very good approximations for these potentials with SWKB doing slightly better than WKB. Further, it will be noticed in table 2 that the SWKB results are identical to the exact ones within the accuracy of the calculation. This raises the intriguing question of whether the SWKB is exact [16] for the potential (28). Unfortunately, the integral involved is too complicated to resolve the question analytically.

Table 1. Exact, WKB and SWKB eigenenergies for the potential (19) with $A = 200$ and $B = 200$. Percentage errors have been calculated to three decimal places to bring out the small errors in the SWKB values.

n	Exact	WKB	Percentage error	SWKB	Percentage error
0	−45.043 72	−44.910 80	−0.295		
1	−35.672 40	−35.555 58	−0.327	−35.672 25	0.000
2	−27.491 64	−27.390 39	−0.368	−27.491 39	−0.001
3	−20.462 21	−20.376 04	−0.421	−20.461 92	−0.001
4	−14.545 56	−14.473 96	−0.492	−14.545 26	−0.002
5	−9.704 08	−9.646 57	−0.593	−9.703 80	−0.003
6	−5.901 64	−5.857 75	−0.744	−5.901 40	−0.004
7	−3.104 98	−3.074 33	−0.987	−3.104 81	−0.006
8	−1.294 62	−1.277 28	−1.339	−1.294 51	−0.008

One can significantly increase the class of CES potentials by using the ideas of supersymmetric quantum mechanics [4]. In particular, for both potentials (19) and (28) one can immediately write down supersymmetry partner potentials $V_+(x) = W^2(x) + W'(x)$ which have the same spectrum except that the ground state is missing. The corresponding eigenfunctions of $V_+(x)$ can be obtained from those of equations (19) and (28) by the formula [4]

$$\psi_n^{(+)}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right)\psi_{n+1}(x) \qquad n = 0, 1, 2, \ldots. \tag{33}$$

Table 2. Exact, WKB and SWKB eigenenergies for the potential (28) with $A = -500$ and $B = 500$. No percentage errors are shown for SWKB values as these are almost identical to the exact values, except for a difference of 1 or 2 in the last significant figure.

n	Exact	WKB	Percentage error	SWKB
0	−617.3312	−617.1961	−0.022	
1	−601.7929	−601.6681	−0.021	−601.7928
2	−587.4669	−587.3521	−0.020	−587.4668
3	−574.3283	−574.2233	−0.018	−574.3282
4	−562.3521	−562.2568	−0.017	−562.3520
5	−551.5138	−551.4278	−0.016	−551.5136
6	−541.7889	−541.7121	−0.014	−541.7887
7	−533.1534	−533.0857	−0.013	−533.1532
8	−525.5838	−525.5248	−0.011	−525.5836
9	−519.0567	−519.0064	−0.010	−519.0565
10	−513.5496	−513.5078	−0.008	−513.5495
11	−509.0405	−509.0069	−0.007	−509.0404
12	−505.5084	−505.4829	−0.005	−505.5083
13	−502.9346	−502.9171	−0.003	−502.9345
14	−501.3113	−501.3018	−0.002	−501.3112

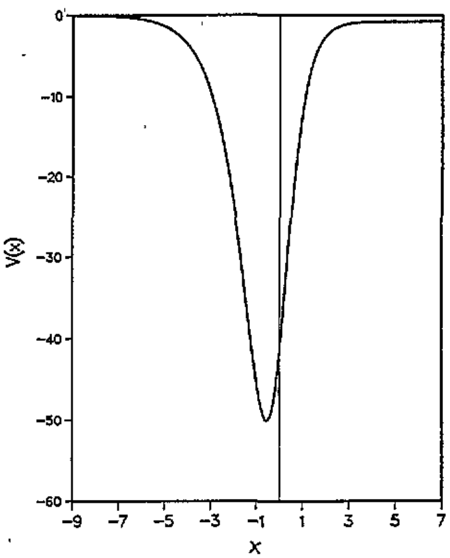

Figure 1. The potential (19) for $A = 200$ and $B = 200$.

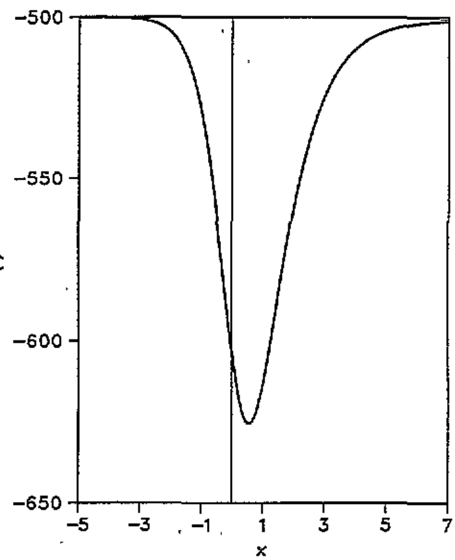

Figure 2. The potential (28) for $A = -500$ and $B = 500$. For the existence of a well in this potential, A should be negative and $B \leqslant |A|$.

Furthermore, corresponding to the two CES potentials (19) and (28), one can also write down one continuous parameter family of potentials with identical spectrum and identical scattering matrix [17] but different (but known) eigenfunctions in terms of those of potentials (19) and (28). These potentials are given by

$$V(x, \lambda) = V(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln(I(x) + \lambda) \tag{34}$$

where $\lambda > 0$ or < -1 but arbitrary otherwise and

$$I(x) = \int_{-\infty}^{x} \psi_0^2(y)\,\mathrm{d}y \tag{35}$$

with ψ_0 being the ground-state-normalized eigenfunction of the potential (19) or (28). All these extensions obviously also apply to the two CES potentials discovered by Dutra [12].

Finally, one could use other ES potentials as V_T and discover further new CES potentials. This is being pursued and will be reported later along with other formal aspects of the CES potentials.

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